

McKay correspondence and tautological bundles on ALE crepant resolutions

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Abstract

We establish a higher dimensional analogue of Kronheimer and Nakajima's geometric McKay correspondence [KN90, Appendix A]. Along the way we prove a rigidity result for certain Hermitian–Yang–Mills metrics on tautological bundles on crepant resolutions of \mathbf{C}^3/G .

1 Introduction

In recent years, the term *McKay correspondence* has come to describe any relation between the group theory of a finite subgroup G of $\mathrm{SL}(n, \mathbf{C})$ and the topology of a crepant resolution of \mathbf{C}^n/G . Its starting point is Du Val's observation [DuV34] that for a finite subgroup G of $\mathrm{SL}(2, \mathbf{C})$ the exceptional divisor E of the unique crepant resolution $\widetilde{\mathbf{C}^2}/G$ of \mathbf{C}^2/G is a tree of rational curves with self-intersection -2 and that these trees are in one-to-one correspondence with the simply-laced Coxeter–Dynkin diagrams. Brieskorn [Bri68] further observed that the intersection matrix of the irreducible components of E is exactly the negative Cartan matrix of the simple Lie algebra associated to the Coxeter–Dynkin tree. This way a correspondence between the finite subgroups of $\mathrm{SL}(2, \mathbf{C})$ and the simply-laced Coxeter–Dynkin diagrams was established. This correspondence was rediscovered from a combinatorial point of view by McKay [McK80], who noticed that for finite subgroups of $\mathrm{SL}(2, \mathbf{C})$ there exists a bijection between the set $\mathrm{Irr}(G)$ of irreducible representations and the set of vertices of the extended Coxeter–Dynkin diagram. To show this, he relates the matrix $A = (a_{\rho\sigma})_{\rho, \sigma \in \mathrm{Irr}(G)}$ corresponding to the decomposition of the tensor products

$$\mathbf{C}^2 \otimes \rho = \bigoplus_{\sigma \in \mathrm{Irr}(G)} a_{\rho\sigma} \sigma$$

to the Cartan matrix \tilde{C} of the extended Coxeter–Dynkin diagram via $A = 2I - \tilde{C}$. These observations together give a complete description of the topology of $\widetilde{\mathbf{C}^2}/G$ in terms of G and its embedding in $\mathrm{SL}(2, \mathbf{C})$, which is now known under the name of the (classical) *McKay correspondence*: The irreducible components Σ_ρ of E are labelled by the non-trivial irreducible representations ρ of G and

$$(1.1) \quad \Sigma_\rho \cdot \Sigma_\sigma = -c_{\rho\sigma}.$$

However, the role of the irreducible representations of G in the above description is still elusive. Hence, it is desirable to find a geometrical interpretation for them.

The first geometric realization of this classical McKay correspondence was given by Gonzalez-Sprinberg and Verdier [GV83] at the level of K -theory. They showed

that the irreducible representation ρ of $G \subset \mathrm{SL}(2, \mathbf{C})$ give rise to holomorphic bundles \mathcal{R}_ρ on $\widetilde{\mathbf{C}^2}/G$, called *tautological bundles*, and these form a basis in K -theory. This way they obtained an isomorphism between the representation ring of G , the G -equivariant K -theory of \mathbf{C}^2 and the K -theory of the crepant resolution. Moreover, they showed that the first Chern classes of the tautological bundles form a basis in cohomology, dual to the basis given by the irreducible components of the exceptional divisor. Using a concrete description of the crepant resolution, Kapranov and Vasserot [KV00] lifted this correspondence to the derived category and showed that it is given by a Fourier–Mukai transform. Yet another interpretation was given by Kronheimer and Nakajima [KN90]. Using the index theorem, they obtained the Cartan matrix as the multiplicative matrix of the first Chern classes of the tautological bundles

$$(1.2) \quad \int_X c_1(\mathcal{R}_\rho) c_1(\mathcal{R}_\sigma) = -(C^{-1})_{\rho\sigma},$$

see [KN90, Theorem A.7].

The string theory insight of Dixon, Harvey, Vafa and Witten [DHVW86] suggested that such McKay correspondences between the topology of a crepant resolution and the representation theory of the finite group should also hold in higher dimension. They introduced the orbifold Euler number, a number expressed in terms of the group G and its action on \mathbf{C}^n , and conjectured that it is equal to the Euler characteristic of any crepant resolution of \mathbf{C}^n/G , whenever such a resolution exists. This has sparked an enormous body of work in mathematics including among other results the case-by-case proof of this conjecture for finite subgroups of $\mathrm{SL}(3, \mathbf{C})$ of Ito [Ito95], Markushevich [Mar97] and Roan [Roa96], its refinement for Betti and Hodge numbers of Batyrev and Dais [BD96], the notion of orbifold cohomology of Chen and Ruan [CR04], and the new versions of the McKay correspondence of Ito and Reid [IR96].

Recall that the existence and uniqueness of a crepant resolution depends crucially on the dimension: In dimension two and three a crepant resolution always exists; however, it is unique only in dimension two, as in dimension three any flop gives another one. In higher dimensions crepant resolutions need not exist at all. Moreover, above dimension two the link between finite groups and Coxeter–Dynkin diagrams is missing. Hence, the insight gained from the geometrical interpretation of the classical McKay correspondence is of key importance to understanding its extensions to higher dimensions. In the case of finite subgroups of $\mathrm{SL}(3, \mathbf{C})$, the K -theory interpretation was carried out by Ito and Nakajima [IN00], while the derived category equivalence was established by Bridgeland, King and Reid [BKR01] and Craw and Ishii [CI04]. All these give additive information about the cohomology of the crepant resolution and they are invariant under its choice. In this work, we obtain the counterpart of Kronheimer and Nakajima’s result and make the first steps towards understanding the multiplicative structure of the cohomology of crepant resolutions of \mathbf{C}^3/G .

We consider the case of a finite subgroup G of $\mathrm{SL}(3, \mathbf{C})$ which acts freely on $\mathbf{C}^3 \setminus \{0\}$. This forces G to be cyclic of prime order. Let X be a projective crepant resolution of \mathbf{C}^3/G . From the work of Craw and Ishii, we know that X is a moduli space \mathcal{M}_θ of G -constellations (a sheaf theoretic generalisation of the notion of G -orbit) which are stable with respect the parameter $\theta \in \Theta_{\mathbf{Q}}$, see Definition 2.3. As such, X naturally comes with a collection of holomorphic bundles \mathcal{R}_ρ , one for each representation ρ of G . The bundles corresponding to the irreducible representations of G form a basis in K -theory, and hence their Chern characters form a basis in $H^*(X, \mathbf{R})$. Using the natural action of G on \mathbf{C}^3 , one can define a matrix C , see (7.5), which is the higher dimensional analogue of the Cartan matrix associated to the finite subgroups of $\mathrm{SL}(2, \mathbf{C})$ from the matrix A .

Theorem 1.3. *Let G be a finite subgroup of $\mathrm{SL}(3, \mathbf{C})$ which acts with an isolated fixed point on \mathbf{C}^3 . Let X be a projective crepant resolution of \mathbf{C}^3/G . Then*

$$(1.4) \quad \int_X \tilde{\mathrm{ch}}(\mathcal{R}_\rho) \wedge \tilde{\mathrm{ch}}(\mathcal{R}_\sigma^*) = - (C^{-1})_{\rho\sigma},$$

for all nontrivial $\rho, \sigma \in \mathrm{Irr}(G)$. Here $\tilde{\mathrm{ch}} := \mathrm{ch} - \mathrm{rk}$ is the reduced Chern character.

Remark 1.5. This result has first appeared in a preprint of the first named author [Deg03], but it had a gap in its proof.

Formula (1.4) should be viewed as an analogue of Kronheimer's and Nakajima's formula (1.2). It determines a part of the multiplicative structure in cohomology of all the projective crepant resolutions of \mathbf{C}^3/G that only depends on the finite group G and its embedding in $\mathrm{SL}(3, \mathbf{C})$. Thus we have found a new McKay correspondence. More precisely, note that the left-hand side of (1.4) can be written as

$$(1.6) \quad \int_X c_1(\mathcal{R}_\rho)^2 c_1(\mathcal{R}_\sigma) - c_1(\mathcal{R}_\rho) c_1(\mathcal{R}_\sigma)^2.$$

Using the Chern-Weil theory, $c_1(\mathcal{R}_\rho) = \frac{1}{2\pi} F_{A_{\theta, \rho}}$ where $F_{A_{\theta, \rho}}$ is the curvature of the natural $(1, 1)$ -connection induced on \mathcal{R}_ρ . By Theorem 3.9, $F_{A_{\theta, \rho}}$ defines a class in the weighted L^2 -cohomology of X , which can be identified, using a result of Hausel, Hunsicker and Mazzeo [HHM04, Corollary 4], with $H_c^k(X, \mathbf{C})$. Moreover, since on the crepant resolution X , we have $H_c^2(X, \mathbf{C}) \cong H^{1,1}(X, \mathbf{C})$, with the last generated by the Poincaré duals of the exceptional divisors, the class corresponding to $c_1(\mathcal{R}_\rho)$ can be represented by compactly supported $(1, 1)$ -forms. Hence, (1.4) describes the part of triple pairing

$$(1.7) \quad \int_X : H^{1,1}(X, \mathbf{C}) \times H^{1,1}(X, \mathbf{C}) \times H^{1,1}(X, \mathbf{C}) \rightarrow \mathbf{C},$$

mapping (α, β, γ) to $\int_X \alpha \wedge \beta \wedge \gamma$ where $\gamma = \alpha - \beta$ and $\alpha, \beta \in \{c_1(\mathcal{R}_\rho) \mid \rho \in \mathrm{Irr}(G)\}$. The issues of completely determining the multiplicative structure in cohomology as well as of describing the entire part which is invariant under the choice of the crepant resolution are still open and will be investigated in future work.

We prove Theorem 1.3 in same manner as Kronheimer and Nakajima by studying the index formula for certain Dirac operators whose index we show to be zero. The condition that the group G acts with an isolated fixed point on \mathbf{C}^3 ensures that any crepant resolution of the orbifold \mathbf{C}^3/G is an ALE space, in which case the index of the Dirac operator is given by the Atiyah–Patodi–Singer (APS) index theorem [APS75]. For all the other finite subgroups of $\mathrm{SL}(3, \mathbf{C})$, the geometry of the crepant resolution is that of a QALE manifold, as introduced by Joyce [Joy00]. The generalization of the APS index theorem to QALE manifolds is work in progress by the first named author [DM12].

In the course of proving Theorem 1.3, we also gain more insight into the geometry of the crepant resolutions $X = \mathcal{M}_\theta$. Since \mathcal{M}_θ is constructed via geometric invariant theory (GIT), it follows that the complex manifold M_θ underlying \mathcal{M}_θ can be constructed as a Kähler quotient. This construction, which we present in Section 3, is the generalisation to higher dimension of Kronheimer's construction of ALE gravitational instantons [Kro89] and was first carried out by Sardo-Infirri [SI96]. One of its consequences is that M_θ carries a natural Kähler metric g_θ . Our second main result establishes the existence of special metrics on M_θ and its tautological bundles \mathcal{R}_ρ .

Theorem 1.8. *Let G be a finite subgroup of $\mathrm{SL}(3, \mathbf{C})$ acting with an isolated fixed point on \mathbf{C}^3 . Let $\theta \in \Theta_{\mathbf{Q}}$ be a generic rational stability parameter. Then*

1. M_θ carries a ALE Ricci-flat Kähler metric $g_{\theta, \text{RF}}$, which is in the same Kähler class as g_θ .
2. for each $\rho \in \text{Irr}(G)$, the holomorphic tautological bundle \mathcal{R}_ρ carries an infinitesimally rigid asymptotically flat Hermitian–Yang–Mills metric.

The existence of the Ricci-flat Kähler metric on M_θ is a consequence of Joyce’s proof of the Calabi conjecture for ALE crepant resolutions [Joy00, Section 8], while the existence of HYM metric is a consequence of the properties of the Laplace operator on ALE manifolds. The difficulty lies in proving the infinitesimal rigidity statement. In fact, the key ingredient for proving both Theorem 1.3 and the rigidity of the HYM metric in Theorem 1.8 is the vanishing result in Lemma 5.1.

The results here are of interest in the context of higher dimensional gauge theory. For example, one can use them to extend the second named author’s construction of G_2 –instantons on generalized Kummer constructions [Wal11] to G_2 –manifolds arising from G_2 –orbifolds with codimension 6 singularities.

The paper is organised as follows. In Section 2 we briefly recall the construction of crepant resolutions as moduli spaces of G –constellations, introduce the Fourier–Mukai transform and collect the results of Bridgeland, King and Reid [BKR01] and Craw and Ishii [CI04] that are relevant for our work. In Section 3 we present the Kähler counterpart of the construction of crepant resolutions and use it to describe its geometry. In Section 4 we prove the existence part of Theorem 1.8, while in Section 5 we prove the rigidity statement. Section 6 introduces the Dirac operator on crepant resolutions and establishes its properties. We finish with the proof of our main Theorem 1.3 in Section 7.

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2 Moduli spaces of G –constellations

Let G be a finite subgroup of $\text{SL}(3, \mathbb{C})$. We denote by $\text{Irr}(G)$ the set of irreducible representations, by $\text{Rep}(G)$ the representation ring, and by R the regular representation of G .

Definition 2.1. A G –sheaf on \mathbb{C}^3 is a coherent sheaf \mathcal{F} together with an action of G which is equivariant with respect to the action of G on \mathbb{C}^3 . A G –constellation on \mathbb{C}^3 is a G –sheaf \mathcal{F} such that $H^0(\mathbb{C}^3, \mathcal{F}) \cong R$ as G –modules. An isomorphism of G –constellations is an isomorphism of sheaves intertwining the G –actions.

Remark 2.2. The (set theoretic) support of a G –constellation is a G –orbit in \mathbb{C}^3 and can thus be thought of as a point in \mathbb{C}^3/G . From this point of view one can think of G –constellations as a sheaf-theoretic generalisation of the notion of G –orbit.

Definition 2.3. The set

$$\Theta := \{\theta \in \text{Hom}_{\mathbb{Z}}(\text{Rep}(G), \mathbb{Z}) : \theta(R) = 0\}$$

is called the *space of integral stability parameters*. The sets $\Theta_{\mathbb{Q}} := \Theta \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\Theta_{\mathbb{R}} := \Theta \otimes_{\mathbb{Z}} \mathbb{R}$ are called the *space of rational stability parameters* and the *space*

of real stability parameters, respectively. Given $\theta \in \Theta_{\mathbf{R}}$, a G -constellation \mathcal{F} is called θ -stable (resp. θ -semi-stable) if each non-trivial proper G -subsheaf $\mathcal{E} \subset \mathcal{F}$ satisfies $\theta(H^0(\mathcal{E})) > 0$ (resp. $\theta(H^0(\mathcal{E})) \geq 0$). A real stability parameter $\theta \in \Theta_{\mathbf{R}}$ is called *generic*, if there is no non-trivial proper subrepresentation $S \subset R$ such that $\theta(S) = 0$.

The space of generic stability parameters is dense in $\Theta_{\mathbf{R}}$. If $\theta \in \Theta_{\mathbf{R}}$ is generic, then every θ -semi-stable G -constellation is θ -stable, c.f. [CI04, Section 2.2].

Theorem 2.4 (Craw and Ishii [CI04, Section 2.1]). *If $\theta \in \Theta_{\mathbf{Q}}$ is generic, then there exists a smooth fine moduli space \mathcal{M}_{θ} of θ -stable G -constellations on \mathbf{C}^3 . Moreover, associated to each representation ρ of G there is a locally free sheaf \mathcal{R}_{ρ} on \mathcal{M}_{θ} . If ρ and σ are two representations of G , then $\mathcal{R}_{\rho \oplus \sigma} = \mathcal{R}_{\rho} \oplus \mathcal{R}_{\sigma}$.*

Sketch of the Proof. The construction of \mathcal{M}_{θ} uses GIT and is based on ideas of King [Kin94] and Sardo-Infrri [SI96].

A G -constellation on \mathbf{C}^3 is a G -equivariant $\mathrm{Sym}^{\bullet}(\mathbf{C}^3)^*$ -module structure on R , i.e., a G -equivariant homomorphism $\mathrm{Sym}^{\bullet}(\mathbf{C}^3)^* \rightarrow \mathrm{End}(R)$. Hence, to each point B in

$$(2.5) \quad N := \left\{ B \in (\mathrm{End}(R) \otimes \mathbf{C}^3)^G : [B \wedge B] = 0 \in \mathrm{End}(R) \otimes \Lambda^2 \mathbf{C}^3 \right\},$$

we can associate the G -constellation defined by $p \in \mathrm{Sym}^{\bullet}(\mathbf{C}^3)^* \mapsto p(B) \in \mathrm{End}(R)$. In fact, every G -constellation arises this way. Furthermore, two points in N yield isomorphic G -constellations if and only if they are related by a G -equivariant automorphism of R , i.e., an element of $\mathrm{GL}(R)^G$. Since $R = \bigoplus_{\rho \in \mathrm{Irr}(G)} \mathbf{C}^{\dim \rho} \otimes \rho$, Schur's lemma gives

$$\mathrm{GL}(R)^G = \prod_{\rho \in \mathrm{Irr}(G)} \mathrm{GL}(\mathbf{C}^{\dim \rho}).$$

Since the diagonal $\mathbf{C}^* \subset \mathrm{GL}(R)^G$ acts trivially on N , the action of $\mathrm{GL}(R)^G$ descends to an action of $\mathbf{PGL}(R)^G$. An integral stability parameter $\theta \in \Theta$ thus determines a character $\chi_{\theta}: \mathbf{PGL}(R)^G \rightarrow \mathbf{C}^*$ defined by

$$(2.6) \quad \chi_{\theta}([g]) = \chi_{\theta}([(g_{\rho})]) := \prod_{\rho \in \mathrm{Irr}(G)} \det(g_{\rho})^{\theta(\rho)}.$$

King [Kin94, Proposition 3.1] proved that an element of N is stable (resp. semi-stable) in the sense of GIT with respect to χ_{θ} if and only if the corresponding G -constellation is θ -stable (resp. θ -semi-stable). Let N_{θ}^s (resp. N_{θ}^{ss}) be the set GIT (semi-)stable points with respect to χ_{θ} in N and let

$$(2.7) \quad \mathcal{M}_{\theta} := N_{\theta}^s / \mathbf{PGL}(R)^G$$

be the corresponding GIT quotient. As schemes, $\mathcal{M}_{k\theta} = \mathcal{M}_{\theta}$ for any $k \in \mathbf{N}$ and therefore, the above construction extends to rational stability parameters $\theta \in \Theta_{\mathbf{Q}}$ as well.

To see that \mathcal{M}_{θ} is indeed a fine moduli space, we construct a *universal G -constellation* \mathcal{U}_{θ} on $\mathcal{M}_{\theta} \times \mathbf{C}^3$. For this, we identify

$$\mathbf{PGL}(R)^G \cong \prod_{\rho \in \mathrm{Irr}_0(G)} \mathrm{GL}(\mathbf{C}^{\dim \rho}),$$

where $\mathrm{Irr}_0(G)$ is the set of *non-trivial* irreducible representations of G . In this way, $\mathbf{PGL}(R)^G$ acts on R . This makes $R \otimes \mathcal{O}_N$ into a $\mathbf{PGL}(R)^G$ -equivariant sheaf

on N . We denote its descend to \mathcal{M}_θ by \mathcal{R} . Since the universal morphism $R \otimes \mathcal{O}_N \rightarrow \mathbf{C}^3 \otimes R \otimes \mathcal{O}_N$ is $\mathbf{PGL}(R)^G$ -equivariant, it descends to a universal morphism $\mathcal{R} \rightarrow \mathbf{C}^3 \otimes \mathcal{R}$ on \mathcal{M}_θ . This determines the universal G -constellation \mathcal{U}_θ on $\mathcal{M}_\theta \times \mathbf{C}^3$. Concretely, \mathcal{U}_θ is the sheaf obtained by pulling back \mathcal{R} to $\mathcal{M}_\theta \times \mathbf{C}^3$ with the action of $\mathcal{O}_{\mathbf{C}^3} = \mathrm{Sym}^\bullet(\mathbf{C}^3)^*$ prescribed by the universal morphism.

Let now $\rho: G \rightarrow \mathrm{Aut}(R_\rho)$ be a representation of G . Since by the above identification $\mathbf{PGL}(R)^G$ acts on R_ρ , the construction that associates to R the sheaf \mathcal{R} can be carried on for R_ρ , giving rise to the sheaf \mathcal{R}_ρ . It is then clear that $\mathcal{R}_{\rho \oplus \sigma} = \mathcal{R}_\rho \oplus \mathcal{R}_\sigma$. \square

To obtain further insight into the spaces \mathcal{M}_θ , it is helpful to make use of the language of derived categories. We first recall the *bounded derived category* $D(\mathcal{A})$ associated with an abelian category \mathcal{A} . For details we refer the reader to Bühler's notes [Büh07], as well as Thomas' article [Tho01] and Huybrechts' book [Huy06], both of which underline the importance of derived categories of coherent sheaves in algebraic geometry. Roughly speaking, $D(\mathcal{A})$ is obtained from the category of bounded chain complexes in \mathcal{A} by formally inverting quasi-isomorphisms. If $A, B \in \mathcal{A}$ are considered as bounded chain complexes concentrated in degree zero, then $\mathrm{Hom}_{D(\mathcal{A})}(A, B)$ is a complex whose cohomology computes $\mathrm{Ext}^\bullet(A, B)$, that is,

$$(2.8) \quad H^\bullet(\mathrm{Hom}_{D(\mathcal{A})}(A, B)) = \mathrm{Ext}^\bullet(A, B).$$

If \mathcal{B} is another abelian category and $f: \mathcal{A} \rightarrow \mathcal{B}$ is a left-exact functor, then one associates to it a *right derived functor* $\mathbf{R}f: D(\mathcal{A}) \rightarrow D(\mathcal{B})$. If $A \in \mathcal{A}$ is considered as a bounded chain complex concentrated in degree zero, then $\mathbf{R}f(A)$ is a complex which computes $R^\bullet f(A)$, that is,

$$(2.9) \quad H^\bullet(\mathbf{R}f(A)) = R^\bullet f(A).$$

A similar construction holds for $g: \mathcal{A} \rightarrow \mathcal{B}$ a right-exact functor, producing a left derived functor $\mathbf{L}g$. When working with derived categories, it is customary to write f and g instead of $\mathbf{R}f$ and $\mathbf{L}g$. We follow this custom in this article.

An important example of such a derived category is $D(\mathrm{Coh}(X))$, the bounded derived category of coherent sheaves $D(\mathrm{Coh}(X))$ over a scheme X . If X and Y are two schemes and $K \in \mathrm{Coh}(X \times Y)$ is a coherent sheaf, then the *Fourier–Mukai transform with kernel K* is the functor $\Phi_K: D(\mathrm{Coh}(X)) \rightarrow D(\mathrm{Coh}(Y))$ defined by

$$\Phi_K(-) := (p_2)_*(p_1^* - \otimes K).$$

Here p_1^* , $(p_2)_*$ and \otimes are taken in the derived sense, with p_1 and p_2 denoting the projections from $X \times Y$ to X and Y respectively. If $f: X \rightarrow Y$ is a morphism and \mathcal{O}_Γ denotes the structure sheaf of its graph $\Gamma \subset X \times Y$, then $\Phi_{\mathcal{O}_\Gamma}$ is f_* .

In our context, we denote by $D(\mathcal{M}_\theta)$ the bounded derived category of coherent sheaves on \mathcal{M}_θ and by $D^G(\mathbf{C}^3)$ the bounded derived category of G -equivariant coherent sheaves on \mathbf{C}^3 , which is the same as the bounded derived category $D([\mathbf{C}^3/G])$ of coherent sheaves on the stack $[\mathbf{C}^3/G]$. One of the key ideas of Bridgeland, King and Reid [BKR01] is to introduce the Fourier–Mukai transform $\Phi_\theta: D(\mathcal{M}_\theta) \rightarrow D^G(\mathbf{C}^3)$ with kernel given by the universal G -constellation \mathcal{U}_θ

$$(2.10) \quad \Phi_\theta(-) = q_*(p^*(- \otimes \rho_0) \otimes \mathcal{U}_\theta)$$

to study crepant resolutions. Here $p: \mathcal{M}_\theta \times \mathbf{C}^3 \rightarrow \mathcal{M}_\theta$ and $q: \mathcal{M}_\theta \times \mathbf{C}^3 \rightarrow \mathbf{C}^3$ are the two projections. Using it we have the following descriptions of the projective crepant resolutions of \mathbf{C}^3/G :

Theorem 2.11 (Craw and Ishii [CI04, Proposition 2.2 and Theorem 2.5]). *For each $\theta \in \Theta_{\mathbf{Q}}$, there exists a morphism $\pi_{\theta}: \mathcal{M}_{\theta} \rightarrow \mathbf{C}^3/G$ which associates to each isomorphism class of G -constellations its support. If $\theta \in \Theta_{\mathbf{Q}}$ is generic, then π_{θ} is a projective crepant resolution and the Fourier-Mukai transform Φ_{θ} is an equivalence of derived categories.*

Remark 2.12. Bridgeland, King and Reid [BKR01] first proved this result for Nakamura's G -Hilbert scheme. Craw and Ishii observed that the proof carries over to the more general moduli spaces of G -constellations. The fact that π_{θ} is a crepant resolution is essentially a consequence of Φ_{θ} being an equivalence of derived category, in which case there exists a categorical criterion for a resolution to be crepant [BKR01, Lemma 3.1].

Theorem 2.13 (Craw and Ishii [CI04, Theorem 1.1]). *If G is an abelian subgroup of $SL(3, \mathbf{C})$, then every projective crepant resolution of \mathbf{C}^3/G is a moduli space of θ -stable G -constellations for some generic $\theta \in \Theta_{\mathbf{Q}}$.*

3 \mathcal{M}_{θ} via Kähler reduction

We now approach the previous discussion from the Kähler point of view. There is no loss in assuming that the finite group $G \subset SL(3, \mathbf{C})$ preserves the standard Hermitian metric on \mathbf{C}^3 , that is, $G \subset SU(3)$. Moreover, we fix a G -invariant Hermitian metric on R . In this set-up, N defined in (2.5) is a cone in the Hermitian vector space $(\text{End}(R) \otimes \mathbf{C}^3)^G$. The space $(\text{End}(R) \otimes \mathbf{C}^3)^G$ is naturally a Kähler manifold with Kähler form

$$\omega(B, C) := \text{Im} \sum_{\alpha=1}^3 \text{tr}(B_{\alpha} C_{\alpha}^*) = \sum_{\alpha=1}^3 \frac{1}{2i} \text{tr}(B_{\alpha} C_{\alpha}^* - B_{\alpha}^* C_{\alpha}).$$

Here we identify $B \in (\text{End}(R) \otimes \mathbf{C}^3)^G$ with a triple (B_1, B_2, B_3) of endomorphisms of R .

Proposition 3.1. *The action of $\text{PU}(R)^G$ on $(\text{End}(R) \otimes \mathbf{C}^3)^G$ by conjugation is Hamiltonian with moment map $\mu: (\text{End}(R) \otimes \mathbf{C}^3)^G \rightarrow (\mathfrak{pu}(R)^G)^*$ given by*

$$\langle \mu(B), \xi \rangle = \sum_{\alpha} \frac{1}{2i} \text{tr}(\xi [B_{\alpha}, B_{\alpha}^*]).$$

Proof. It is enough to prove this for the action of $U(R)^G$. If $\xi \in \mathfrak{u}(R)^G$, then the corresponding vector field X_{ξ} on $(\text{End}(R) \otimes \mathbf{C}^3)^G$ is given by $X_{\xi}(B) = [\xi, B]$. Thus

$$\begin{aligned} i(X_{\xi})\omega(\hat{B}) &= \sum_{\alpha} \frac{1}{2i} \text{tr} \left([\xi, B_{\alpha}] \hat{B}_{\alpha}^* - [\xi, B_{\alpha}]^* \hat{B}_{\alpha} \right) \\ &= \sum_{\alpha} \frac{1}{2i} \text{tr} \left(\xi ([B_{\alpha}, \hat{B}_{\alpha}^*] + [\hat{B}_{\alpha}, B_{\alpha}^*]) \right) = \langle d\mu(B) \hat{B}, \xi \rangle. \end{aligned}$$

□

For $\theta \in \Theta_{\mathbf{R}}$, we define $\zeta_{\theta} \in (\mathfrak{pu}(R)^G)^*$ by

$$(3.2) \quad \zeta_{\theta}(\xi) := - \sum_{\rho \in \text{Irr}(G)} i\theta(\rho) \text{tr}(\xi \cdot \pi_{\rho})$$

for all $\xi \in \mathfrak{pu}(R)^G$. Here $\pi_{\rho}: R \rightarrow \mathbf{C}^{\dim \rho} \otimes R_{\rho}$ is the projection onto the ρ -isotypical component of the regular representation, and $\xi \cdot \pi_{\rho}$ is thought of as an element

in $\text{End}(R)$. Note that if θ is integral, then $\zeta_\theta = -\text{id}\chi_\theta \in (\mathfrak{pu}(R)^G)^*$, with χ_θ the character associated to θ as defined in (2.6). Moreover, since the centre of $\mathfrak{u}(R)^G$ is spanned by $\{i\pi_\rho \mid \rho \in \text{Irr}(G)\}$, we can identify $\Theta_{\mathbf{R}}$ with the centre of $(\mathfrak{pu}(R)^G)^*$ via $\theta \mapsto \zeta_\theta$.

With this identification, it follows that $\theta \in \Theta_{\mathbf{R}}$ is generic if and only if for all proper subrepresentations $0 \subsetneq S \subsetneq R$ we have $\zeta_\theta(i\pi_S) \neq 0$, with $\pi_S: R \rightarrow S$ denoting the orthogonal projection onto S .

For each $\theta \in \Theta_{\mathbf{R}}$, we denote by

$$(3.3) \quad M_\theta := \mu^{-1}(\zeta_\theta)/\mathbf{PU}(R)^G$$

corresponding Kähler quotient. Moreover, for each representation $\rho: G \rightarrow \text{GL}(R_\rho)$ of G , $\mathbf{PU}(R)^G$ acts on R_ρ and gives rise to the bundle

$$\mathcal{R}_\rho = \mu^{-1}(\theta) \times_{\mathbf{PU}(R)^G} R_\rho$$

on M_θ . This bundle is holomorphic, since the holomorphic structure on $\mu^{-1}(\theta) \times V$ is $\mathbf{PU}(R)^G$ -equivariant and thus passes down to \mathcal{R}_ρ . We call \mathcal{R}_ρ the *tautological holomorphic bundle* associated to the representation ρ of G .

We describe now the relation between the algebraic objects \mathcal{M}_θ and \mathcal{R}_ρ defined in Theorem 2.4 and the holomorphic objects M_θ and \mathcal{R}_ρ defined above.

Proposition 3.4. *If $B \in N \cap \mu^{-1}(\zeta_\theta)$, then the G -sheaf \mathcal{F} associated to B is θ -semi-stable. Therefore we have $\mu^{-1}(\zeta_\theta) \subset N_\theta^{ss}$ for all $\theta \in \Theta_{\mathbf{R}}$.*

Proof. Let \mathcal{E} be a non-trivial proper G -subsheaf of \mathcal{F} . Then we can decompose the regular representation into two non-trivial proper subrepresentations

$$R = S \oplus T$$

with $S := H^0(\mathcal{E})$ and T its orthogonal complement. Corresponding to \mathcal{E} there is a triple of matrices $C \in \text{End}(S) \otimes \mathbf{C}^3$. Moreover, since each component of B leaves S invariant, we have

$$B = \begin{pmatrix} C & D \\ 0 & E \end{pmatrix}.$$

With this,

$$\langle \mu(B), i\pi_S \rangle = \frac{1}{2} \text{tr}_S ([C, C^*] + DD^*) = \frac{1}{2} \text{tr}_S (DD^*) \geq 0.$$

Since $\langle \mu(B), i\pi_S \rangle = \theta(i\pi_S) = \theta(S)$, it follows that $\theta(H^0(\mathcal{E})) \geq 0$. \square

King [Kin94, Theorem 6. 1] shows that the Kempf–Ness theorem holds in this case: If $\theta \in \Theta$, then each $\mathbf{PGL}(R)^G$ -orbit which is closed in N_θ^{ss} meets $\mu^{-1}(\zeta_\theta)$ in precisely one $\mathbf{PU}(R)^G$ -orbit, and meets no other orbit. From this we have the following result:

Proposition 3.5. *Suppose $\theta \in \Theta_{\mathbf{Q}}$ is generic. Then the inclusion $\mu^{-1}(\zeta_\theta) \rightarrow N_\theta^s = N_\theta^{ss}$ induces a biholomorphic map from M_θ to the analytification of \mathcal{M}_θ . This map identifies the holomorphic bundle \mathcal{R}_ρ with the analytification of the locally free sheaf \mathcal{R}_ρ .*

Now, for each $\theta \in \Theta_R$, let g_θ and ω_θ be the metric and the Kähler form on M_θ induced by the Kähler quotient construction. We also have a natural $\mathbf{PU}(R)^G$ -connection A_θ whose horizontal space is the orthogonal complement of the tangent space to the orbit in $T_B(\mu^{-1}(\theta))$. To describe the geometry of M_θ and the behaviour of g_θ and A_θ , we first need the following definitions:

Definition 3.6. Let G be a finite subgroup of $SU(3)$ acting freely on $\mathbf{C}^3 \setminus \{0\}$. A noncompact Riemannian manifold (X, g) of real dimension 6 is called an *ALE manifold asymptotic to \mathbf{C}^3/G to order $\tau > 0$* , if there exists a compact subset $K \subset X$ and a diffeomorphism $\pi: (\mathbf{C}^3 \setminus B_R)/G \rightarrow X \setminus K$ so that

$$\partial_k(\pi^*g - g_0) = O(r^{-\tau-k})$$

for all $0 \leq k \leq 2$. Here $r := |x|$ denotes the radius function on \mathbf{C}^3 and g_0 denotes the standard metric on \mathbf{C}^3 . The pair $(X \setminus K, \pi)$ is called an *ALE end* and the function r , extended smoothly to X so that $r: X \rightarrow [1, \infty)$, is a *radius function* on X .

Definition 3.7. Similarly, a connection A on a complex vector bundle E of rank k on the ALE manifold (X, g) is called *asymptotically flat of order $\tau > 0$* if there exists a flat connection A_0 on the ALE end of X so that

$$\nabla^k(A - A_0) = O(r^{-\tau-k})$$

for $0 \leq k \leq 1$.

Definition 3.8. A Hermitian metric h on a complex bundle E on an ALE manifold (X, g) is called *asymptotically flat of order $\tau > 0$* if with respect a local trivialization of E on the ALE end of X it satisfies

$$\partial_k(h - h_0) = O(r^{-\tau-k})$$

for all $0 \leq k \leq 2$, where h_0 denotes the standard metric.

Theorem 3.9. *Let G be a finite subgroup of $SU(3)$ acting feely on $\mathbf{C}^3 \setminus \{0\}$. Then the following holds:*

1. M_0 is isometric to the orbifold \mathbf{C}^3/G with the induced orbifold Kähler metric g_0 , and the connection A_0 is flat.
2. If $\theta \in \Theta_{\mathbf{Q}}$ generic, then M_θ is smooth and the induced Kähler metric g_θ is ALE of order 4.
3. The $\mathbf{PU}(R)^G$ -connection A_θ is an $(1, 1)$ -connection which is asymptotically flat of order 2. In particular, its curvature decays like r^{-4} .

In the case of finite subgroups of $SU(2)$, the analogous theorem was proven by Kronheimer [Kro89] and by Gocho and Nakajima [GN92]. For the above theorem, the smoothness of the Kähler quotient M_θ for $\theta \in \Theta_{\mathbf{Q}}$ generic follows from the identification with the algebraic quotient \mathcal{M}_θ provided by Proposition 3.5 and the result of Theorem 2.11. The first statement and the remaining of the second were proved by Sardo-Infirri [SI96] by generalising Kronheimer's proof. The proof of the third statement is a direct generalization of Gocho and Nakajima's argument.

Remark 3.10. It seems reasonable to expect that the second statement in Theorem 3.9 holds for all generic $\theta \in \Theta_{\mathbf{R}}$. Because of the homogeneity of the moment map, the statement follows for all $t\theta$ with $\theta \in \Theta_{\mathbf{Q}}$ generic and t a strictly positive real number. On the other hand, when $\theta \in \Theta_{\mathbf{R}}$ generic, we can show that $\mathbf{PU}(R)^G$ acts freely on $\mu^{-1}(\theta)$. However, to conclude that M_θ is smooth, one still needs to show that $\mu^{-1}(\theta)$ is contained in the smooth locus of N .

4 Ricci-flat metrics on M_θ and Hermitian–Yang–Mills metrics on \mathcal{R}_ρ

In contrast to Kronheimer’s and Nakajima’s work [Kro89, KN90], the metric g_θ on M_θ obtained from the Kähler reduction is not necessarily Ricci-flat and the connections A_ρ on \mathcal{R}_ρ are not necessarily Hermitian–Yang–Mills (HYM). Indeed, Sardo-Infirri [SI96, Example 7.1] showed that for the finite subgroup $G = \mathbf{Z}_3$ and an appropriate choice of θ generic, the induced metric on $M_\theta = \mathcal{O}_{\mathbf{CP}^2}(-3)$ has non-vanishing Ricci curvature. In this section, we show that M_θ does admit a Ricci-flat Kähler metric and \mathcal{R}_ρ carries an asymptotically flat HYM connection, thus partially proving Theorem 1.8. The existence of the Ricci-flat Kähler metric follows from the following result:

Theorem 4.1 (Joyce [Joy00, Theorem 8.2.3]). *Let G be a finite subgroup of $\mathrm{SU}(n)$ acting freely on $\mathbf{C}^n \setminus \{0\}$. Let X be a smooth crepant resolution of \mathbf{C}^n/G with an ALE Kähler metric g of order $\tau > n$. Then there exists a unique Ricci-flat ALE Kähler metric g_{RF} in the Kähler class of g . The metric g_{RF} is ALE of order $2n$.*

Remark 4.2. Joyce states this result only for ALE Kähler metrics of order $\tau = 2n$; however, his proof goes through for $\tau > n$.

Corollary 4.3. *Let $\theta \in \Theta_{\mathbf{Q}}$ generic and (M_θ, g_θ) the corresponding Kähler quotient. Then there exists an ALE Kähler Ricci-flat metric $g_{\theta, \mathrm{RF}}$ of order 6 on M_θ in the same Kähler class as g_θ .*

Proposition 4.4. *Let X be an ALE Kähler manifold and let \mathcal{L} be a holomorphic line bundle on X . If there is a Hermitian metric h_0 on \mathcal{L} such that*

$$\Lambda F_{h_0} = O(r^{-2-\varepsilon})$$

for some $\varepsilon > 0$, then there exists an $\delta > 0$ and a HYM metric h on \mathcal{L} asymptotic to h_0 to order δ . Moreover, for any $\gamma > 0$, h is the unique HYM metric on \mathcal{L} asymptotic to h_0 to order γ .

Corollary 4.5. *Let $\theta \in \Theta_{\mathbf{Q}}$ generic. Then for each $\rho \in \mathrm{Irr}(G)$ the tautological bundle \mathcal{R}_ρ on M_θ carries an asymptotically flat HYM metric with respect to $g_{\theta, \mathrm{RF}}$.*

Remark 4.6. Using some of the results derived in Section 5, one can show that the HYM connection associated with h in Proposition 4.4 is asymptotically flat of order 5. Thus the Hermitian metric h is asymptotically flat of order 4.

Remark 4.7. Using heat flow methods, Bando [Ban93] proved that every holomorphic bundle \mathcal{E} over an ALE Kähler manifold which admits a Hermitian metric h_0 with $|F_{h_0}| = O(r^{-2-\varepsilon})$ does in fact carry a HYM metric.

The case of line bundles is much simpler and follows from the Laplace operator being an isomorphism between certain weighted Sobolev spaces. For k a nonnegative integer and $\delta \in \mathbf{R}$, we denote by $W_\delta^{k,2}(X)$ the completion of $C_0^\infty(X)$ with respect to the norm

$$(4.8) \quad \|f\|_{W_\delta^{k,2}} := \sum_{j=0}^k \|r^{-\delta-m/2+j} \nabla^j f\|_{L^2}.$$

Here m denotes the real dimension of X . Let $\Delta_\delta: W_\delta^{k+2,2}(X) \rightarrow W_{\delta-2}^{k,2}(X)$ denote the corresponding completion of the Laplacian Δ .

Proposition 4.9. *For $\delta \in (-m+2, 0)$, the operator Δ_δ is an isomorphism.*

Proof. The weighted Laplacian $\Delta_\delta: W_\delta^{k+2,2}(X) \rightarrow W_{\delta-2}^{k,2}(X)$ is a Fredholm operator if and only if the weight parameter δ is not in its set of indicial roots at infinity. This is a discrete set that does not intersect the interval $(-m+2, 0)$, see Bartnik [Bar86, Sections 1 and 2] for details. Moreover, for $\delta < 0$, the kernel of Δ_δ is trivial by the maximum principle. On the other hand, the cokernel of Δ_δ is isomorphic to the kernel of its formal adjoint $\Delta_{m-2-\delta}$. Therefore, for $\delta \in (-m+2, 0)$, Δ_δ is an isomorphism. \square

Proof of Proposition 4.4. Any Hermitian metric on \mathcal{L} is of the form $h = e^f h_0$, for $f \in C^\infty(X)$. Then $F_h = F_{h_0} + \bar{\partial}\partial f \in \Omega^2(X, i\mathbf{R})$ and thus

$$i\Lambda F_h = i\Lambda F_{h_0} + \frac{1}{2}\Delta f.$$

Since $\Lambda F_{h_0} \in W_{-2-\varepsilon}^{0,2}(X)$ for some $\varepsilon > 0$, by Proposition 4.9, there exists a unique f such that $\Delta f = -2i\Lambda F_{h_0}$ and $f = O(r^{-\delta})$ for some $\delta > 0$. \square

5 Rigidity of HYM metrics on the holomorphic tautological bundles

In this section we prove the rigidity statement in Theorem 1.8. This will be an immediate consequence of Lemma 5.1, which is the main vanishing result of this paper.

Lemma 5.1. *Let $\theta \in \Theta_{\mathbf{Q}}$ generic and let M_θ be equipped with an ALE Kähler metric. Let h be a Hermitian metric on \mathcal{R} and let A denote the associated Chern connection. Suppose that A is asymptotically flat of order $\tau > 0$. Then the space*

$$\mathcal{H}_A^1 := \left\{ a \in \Omega^{0,1}(M_\theta, \text{End}(\mathcal{R})) : \bar{\partial}_A a = \bar{\partial}_A^* a = 0 \text{ and } \lim_{r \rightarrow \infty} \sup_{\partial B_r} |a| = 0 \right\}$$

is trivial.

If the metric h is HYM, then \mathcal{H}_A^1 is the space of infinitesimal deformations. Hence, the HYM metrics on \mathcal{R}_ρ constructed in the first part of Theorem 1.8 are infinitesimally rigid for all $\rho \in \text{Irr}(G)$, thus completing the proof of Theorem 1.8.

The strategy for proving Lemma 5.1 is as follows: We first reduce to a problem in complex geometry, see Proposition 5.8. Using GAGA, we translate this into an algebraic geometry problem, see (5.9), which we then solve using the results of Bridgeland, King and Reid [BKR01] and Craw and Ishii [CI04] discussed in Section 2.

It is a useful heuristic to think of bundles with decaying connections as bundles on a compactification whose restrictions to the “divisor at infinity” satisfy certain conditions, like being flat for example. Accordingly, we compactify M_θ at infinity by gluing M_θ and $(\mathbf{P}^3 \setminus \{[0:0:0:1]\})/G$ along $M_\theta \setminus \pi_\theta^{-1}(0) = (\mathbf{C}^3 \setminus \{0\})/G$. The resulting space \bar{M}_θ is not a complex manifold, but rather a complex orbifold. We denote its divisor at infinity by D . This is a smooth orbifold divisor, i.e., it lifts to a smooth divisors in covers of the uniformising charts. The bundle \mathcal{R} extends over D to a bundle $\bar{\mathcal{R}}$ on \bar{M}_θ . The following result reduces the proof of Lemma 5.1 to a problem in complex geometry.

Proposition 5.2. \mathcal{H}_A^1 injects into $H^1(\bar{M}_\theta, \text{End}(\bar{\mathcal{R}})(-D))$.

The proof of Proposition 5.2 requires two preparatory results.

Proposition 5.3. *Let Z be a complex orbifold, D be a smooth divisor in Z and \mathcal{E} be a holomorphic bundle on Z . Denote by $i: D \hookrightarrow Z$ the inclusion of D into Z . Then the complex of sheaves $(\mathcal{A}^\bullet, \bar{\partial})$ defined by*

$$\mathcal{A}^k(U) := \{\alpha \in \Omega^{0,k}(U, \mathcal{E}) : i^* \alpha = 0\}$$

for $U \subset Z$ open is an acyclic resolution of $\mathcal{E}(-D)$.

Proof. Since the i^* and $\bar{\partial}$ commute, \mathcal{A}^\bullet forms a complex. Moreover, it is clear that $\mathcal{E}(-D)$ is the kernel of $\mathcal{A}^0 \xrightarrow{\bar{\partial}} \mathcal{A}^1$.

The proof that \mathcal{A}^\bullet is a resolution uses two ingredients: the Grothendieck–Dolbeault Lemma and the fact that if U is a sufficiently small open set, then holomorphic sections on $D \cap U$ extend to U . We show that these assertions hold also for orbifolds. Let U be a small open set which is covered by a uniformising chart \tilde{U}/Γ . Lifting everything up to \tilde{U} , \mathcal{E} corresponds to a Γ -equivariant holomorphic bundle $\tilde{\mathcal{E}}$ and D to a Γ -equivariant smooth divisor \tilde{D} . If $\alpha \in \Omega^{(0,k)}(U, \mathcal{E})$ satisfies $\bar{\partial}\alpha = 0$, then so does its lift $\tilde{\alpha} \in \Omega^{(0,k)}(\tilde{U}, \tilde{\mathcal{E}})^\Gamma$. If U (and thus \tilde{U}) is sufficiently small, then the usual Grothendieck–Dolbeault Lemma yields $\tilde{\beta} \in \Omega^{(0,k-1)}(\tilde{U}, \tilde{\mathcal{E}})$ satisfying $\bar{\partial}\tilde{\beta} = \tilde{\alpha}$. There is no loss in assuming that $\tilde{\beta}$ is Γ -invariant and thus pushes down to the desired primitive $\beta \in \Omega^{(0,k-1)}(U, \mathcal{E})$ of α . We thus obtain the Grothendieck–Dolbeault Lemma for orbifolds. Now, if s is a holomorphic section of \mathcal{E} over $D \cap U$, we lift it to the uniformising chart \tilde{U} , where, provided U is sufficiently small, we find a Γ -equivariant extension. We then push this extension down to U . Hence, the proof of the second assertion in orbifold set-up.

Let now U be a small open set of Z and let $\alpha \in \Omega^{0,k}(U, \mathcal{E})$ with $\bar{\partial}\alpha = 0$. By the Grothendieck–Dolbeault Lemma after possibly shrinking U , we can find $\beta \in \Omega^{0,k}(U, \mathcal{E})$ satisfying $\bar{\partial}\beta = \alpha$. If $k \geq 2$, we apply the Grothendieck–Dolbeault Lemma once more to obtain $\gamma \in \Omega^{0,k-2}(U \cap D, \mathcal{E})$ such that $\bar{\partial}\gamma = i^*\beta$. We extend γ smoothly to all of U . Then $\beta - \bar{\partial}\gamma \in \mathcal{A}^{k-1}(U)$ yields the desired primitive of α on U . When $k = 1$, we know that β restricts to a holomorphic section $\beta|_D$ of $\mathcal{E}|_{U \cap D}$, which can be extended to a holomorphic section δ on U . Hence, $\beta - \delta \in \mathcal{A}^0(U)$ is the desired primitive of α .

Finally, (\mathcal{A}^\bullet, d) is an acyclic resolution of $\mathcal{E}(-D)$, since the sheaves \mathcal{A}^\bullet are C^∞ -modules and therefore soft. \square

Remark 5.4. In the definition of \mathcal{A}^k it is not strictly necessary to require that α be smooth. In fact, a simple application of elliptic regularity shows that it suffices that elements of \mathcal{A}^k be in the Hölder space $C^{n-k,\alpha}$, where n denote the complex dimension of Z .

Proposition 5.5. *If $a \in \mathcal{H}_A^1$, then*

$$(5.6) \quad \nabla^k a = O(r^{-5-k})$$

for all $k \geq 0$.

Proof. First observe that using simple scaling considerations and standard elliptic theory, (5.6) for $k > 0$ follows from the case $k = 0$.

It is rather straightforward to obtain $a = O(r^{-4})$ using the maximum principle. To obtain the stronger decay estimate it is customary to make use of a refined Kato inequality, see, e.g., Bando, Kasue and Nakajima [BKN89]. The Kato inequality is a consequence of the following application of the Cauchy-Schwarz inequality: $|\langle \nabla_A a, a \rangle| \leq |\nabla a| |a|$. But, the equation $\bar{\partial}a = \bar{\partial}_A^* a = 0$ imposes a linear constraint on $\nabla_A a$, which is incompatible with equality in the previous estimate unless $\nabla_A a = 0$. Hence, there exists a constant $\gamma < 1$, such that $|d|a| \leq \gamma |\nabla_A a|$ on the set $U := \{x \in M_\theta : a(x) \neq 0\}$. A more detailed analysis shows that γ can be chosen to

be $\sqrt{5/6}$. For a systematic treatment of refined Kato inequalities we refer to the work of Calderbank, Gauduchon and Herzlich [CGH00].

We set $\gamma = \sqrt{5/6}$ and let $\sigma = 2 - 1/\gamma^2 = 4/5$. Using the refined Kato inequality for a , we have

$$\begin{aligned} (2/\sigma)\Delta|a|^\sigma &= |a|^{\sigma-2} (\Delta|a|^2 - 2(\sigma-2)|d|a||^2) \\ &\leq |a|^{\sigma-2} (\Delta|a|^2 + 2|\nabla_A a|^2) \\ &= |a|^{\sigma-2} \langle a, \nabla_A^* \nabla_A a \rangle. \end{aligned}$$

The Weitzenböck formula for $\nabla_A^* \nabla_A a$ gives

$$(2/\sigma)\Delta|a|^\sigma = |a|^{\sigma-2} (\langle \Delta_{\bar{\partial}} a, a \rangle + \langle \{R, a\}, a \rangle + \langle \{F_A, a\}, a \rangle),$$

with R the Riemannian curvature operator and F_A the curvature of the connection A . Since $\Delta_{\bar{\partial}} a = 0$, and since by Theorem 3.9 the metric on M_θ is ALE of order 4 and the curvature F_A decays like r^{-4} , it follows that there exist positive constants $c, \tau > 0$ so that on U we have

$$(2/\sigma)\Delta|a|^\sigma \leq cr^{-2-\tau}|a|^\sigma.$$

Set $f = |a|^\sigma$. We show that $f = O(r^{-4})$, which is equivalent to the desired decay estimate for a . Note that on U ,

$$(5.7) \quad \Delta f \leq \frac{cf}{1 + r^{2+\tau}}.$$

Since f is bounded, using [Joy00, Theorem 8.3.6(a)], we find $g = O(r^{-\tau})$, such that

$$\Delta g = \begin{cases} (\Delta f)^+ & \text{on } U \\ 0 & \text{on } M_\theta \setminus U \end{cases}$$

Here $(-)^+$ denotes taking the positive part. Then $f - g$ is a subharmonic function on M_θ and must achieve its maximum at the boundary boundary of U or at infinity. Hence $f \leq g = O(r^{-\tau})$. Then by (5.7), $\Delta f = O(r^{-2-2\tau})$ and the above procedure yields $f = O(r^{-2\tau})$. Reiterating this argument k -times gives $f = O(r^{-k\tau})$ for all $k < (n-2)/\tau$. For the biggest k with this property, we have $2 + (k+1)\tau > n$. Then by [Joy00, Theorem 8.3.6(b)], we can chose g above such that $g = O(r^{-4})$. Therefore, $f = O(r^{-4})$ as desired. \square

Proof of Proposition 5.2. Given $a \in \mathcal{H}_A^1$, we extend it to a 1-form on \bar{M}_θ vanishing along D . From Proposition 5.5 it follows that a vanishes to third order along D . Hence, a is in $C^{2,\alpha}$ and we can regard it as an element of $\mathcal{A}^1(\bar{M}_\theta)$. Since $\bar{\partial}a = 0$, by Proposition 5.3 it gives an element $[a] \in H^1(\bar{M}_\theta, \mathcal{E}nd(\bar{\mathcal{R}})(-D))$. This defines a linear map $i: \mathcal{H}_A^1 \rightarrow H^1(\bar{M}_\theta, \mathcal{E}nd(\bar{\mathcal{R}})(-D))$.

We show now that i is injective. For this, assume that there exists $b \in \mathcal{A}^0(\bar{M}_\theta)$ so that $a = \bar{\partial}b$. Since b vanishes along D , its restriction to M_θ decays like r^{-1} . Using this together with $a = O(r^{-5})$, we can integrate by parts to obtain

$$\|a\|_{L^2}^2 = \int_{M_\theta} \langle a, \bar{\partial}b \rangle \, \text{dvol}(g_\theta) = \int_{M_\theta} \langle \bar{\partial}_A^* a, b \rangle \, \text{dvol}(g_\theta) = 0.$$

It follows that a vanishes, and thus i is injective. \square

To prove Lemma 5.1 it now suffices to establish the following result:

Proposition 5.8. $H^1(\bar{M}_\theta, \mathcal{E}nd(\bar{\mathcal{R}})(-D)) = 0$.

To prove this statement, we convert it into an algebraic geometry problem. In the same way we compactified M_θ , we can complete the scheme \mathcal{M}_θ at infinity by attaching $D = \mathbf{P}^2/G$. This yields an algebraic stack $\bar{\mathfrak{M}}_\theta$. Moreover, \mathcal{R} extends to a locally free sheaf $\bar{\mathcal{R}}$ on $\bar{\mathfrak{M}}_\theta$. Using GAGA [Toe99, Théorème 5.10], we see that Proposition 5.8 is equivalent to

$$(5.9) \quad H^1(\bar{\mathfrak{M}}_\theta, \mathcal{E}nd(\bar{\mathcal{R}})(-D)) = 0.$$

To show this, we use the following consequence of Theorem 2.11:

Proposition 5.10. *For $\theta \in \Theta_{\mathbf{Q}}$ generic,*

$$H^k(\mathcal{M}_\theta, \mathcal{R}_\rho^* \otimes \mathcal{R}_\sigma) = H^k(\mathbf{C}^3, \mathcal{O} \otimes R_\rho^* \otimes R_\sigma)^G,$$

for all $\rho, \sigma \in \text{Irr}(G)$. In particular, for $k > 0$,

$$H^k(\mathcal{M}_\theta, \mathcal{R}_\rho^* \otimes \mathcal{R}_\sigma) = 0.$$

If G acts freely on $\mathbf{C}^3 \setminus \{0\}$, we have a commutative diagram

$$\begin{array}{ccc} H^k(\mathcal{M}_\theta, \mathcal{E}nd(\mathcal{R})) & \xrightarrow{i^*} & H^k(\mathcal{M}_\theta \setminus \pi_\theta^{-1}(0), \mathcal{E}nd(\mathcal{R})) \\ \downarrow \Phi_\theta & & \downarrow (\pi_\theta)_* \\ H^k(\mathbf{C}^3, \mathcal{O} \otimes \text{End}(R))^G & \xrightarrow{j^*} & H^k((\mathbf{C}^3 \setminus \{0\})/G, \mathcal{O} \otimes \text{End}(R)), \end{array}$$

where $i: \mathcal{M}_\theta \setminus \pi_\theta^{-1}(0) \rightarrow \mathcal{M}_\theta$ and $j: \mathbf{C}^3 \setminus \{0\} \rightarrow \mathbf{C}^3$ are the inclusion maps.

Proof. The first part is due to Craw and Ishii [CI04, Lemma 5.4]. Let us briefly recall their proof: On the one hand, we have

$$\begin{aligned} H^k(\mathcal{M}_\theta, \mathcal{R}_\rho^* \otimes \mathcal{R}_\sigma) &= \text{Ext}^k(\mathcal{O}, \mathcal{R}_\rho^* \otimes \mathcal{R}_\sigma) = \text{Ext}^k(\mathcal{R}_\rho, \mathcal{R}_\sigma) \\ &= H^k(\text{Hom}_{D(\mathcal{M}_\theta)}(\mathcal{R}_\rho, \mathcal{R}_\sigma)) \end{aligned}$$

and

$$\begin{aligned} H^k(\mathbf{C}^3, \mathcal{O} \otimes R_\rho^* \otimes R_\sigma)^G &= G\text{-Ext}^k(\mathcal{O} \otimes R_\rho, \mathcal{O} \otimes R_\sigma) \\ &= H^k(\text{Hom}_{D^G(\mathbf{C}^3)}(\mathcal{O} \otimes R_\rho, \mathcal{O} \otimes R_\sigma)). \end{aligned}$$

On the other hand, the inverse of the Fourier–Mukai transform Φ_θ is given by

$$\Phi_\theta^{-1} = (p_*(q^*(-) \otimes \mathcal{U}_\theta^D[3]))^G = \left(- \otimes \bigoplus_{\rho} \mathcal{R}_\rho^* \otimes R_\rho \right)^G,$$

see [CI04, p. 267]. Here $(-)^D$ denotes the derived dual. In particular,

$$(5.11) \quad \Phi_\theta^{-1}(\mathcal{O} \otimes R_\rho) = \mathcal{R}_\rho.$$

Then, according to Theorem 2.11, $H^k(\text{Hom}_{D(\mathcal{M}_\theta)}(\mathcal{R}_\rho, \mathcal{R}_\sigma)) \cong H^k(\text{Hom}_{D^G(\mathbf{C}^3)}(\mathcal{O} \otimes R_\rho, \mathcal{O} \otimes R_\sigma))$. Therefore, $H^k(\mathcal{M}_\theta, \mathcal{R}_\rho^* \otimes \mathcal{R}_\sigma) \cong H^k(\mathbf{C}^3, \mathcal{O} \otimes R_\rho^* \otimes R_\sigma)^G$.

To prove the second part, we show that the following diagram commutes

$$\begin{array}{ccc} D(\mathcal{M}_\theta) & \xrightarrow{i^*} & D(\mathcal{M}_\theta \setminus \pi_\theta^{-1}(0)) \\ \downarrow \Phi_\theta & & \downarrow \Phi_{\sigma_\Gamma} \\ D^G(\mathbf{C}^3) & \xrightarrow{j^*} & D^G(\mathbf{C}^3 \setminus \{0\}). \end{array}$$

Here $\Phi_{\mathcal{O}_\Gamma}$ is the Fourier–Mukai transform with kernel \mathcal{O}_Γ , the structure sheaf of the graph of $\pi_\theta: \mathcal{M}_\theta \setminus \pi_\theta^{-1}(0) \rightarrow \mathbf{C}^3 \setminus \{0\}$. Note that under the identification $D^G(\mathbf{C}^3 \setminus \{0\}) = D((\mathbf{C}^3 \setminus \{0\})/G)$, $\Phi_{\mathcal{O}_\Gamma}$ becomes $(\pi_\theta)_*$.

Denote by r and s the projections from $\mathcal{M}_\theta \setminus \pi_\theta^{-1}(0) \times \mathbf{C}^3 \setminus \{0\}$ to $\mathcal{M}_\theta \setminus \pi_\theta^{-1}(0)$ and $\mathbf{C}^3 \setminus \{0\}$, respectively. Let $t: \mathcal{M}_\theta \times \mathbf{C}^3 \setminus \{0\} \rightarrow \mathbf{C}^3 \setminus \{0\}$ denote the projection onto the second factor. The following diagram summarises the situation:

$$\begin{array}{ccccc}
& \mathcal{M}_\theta \setminus \pi_\theta^{-1}(0) \times \mathbf{C}^3 \setminus \{0\} & & & \\
& \swarrow r & \downarrow i \times \text{id} & \searrow s & \\
\mathcal{M}_\theta \setminus \pi_\theta^{-1}(0) & & \mathcal{M}_\theta \times \mathbf{C}^3 \setminus \{0\} & \xrightarrow{t} & \mathbf{C}^3 \setminus \{0\} \\
\downarrow i & & \downarrow \text{id} \times j & & \downarrow j \\
\mathcal{M}_\theta & \xleftarrow{p} & \mathcal{M}_\theta \times \mathbf{C}^3 & \xrightarrow{q} & \mathbf{C}^3
\end{array}$$

It follows, essentially from the definition of π_θ , that

$$(5.12) \quad (\text{id}_{\mathcal{M}_\theta} \times j)^* \mathcal{U}_\theta = (i \times \text{id}_{\mathbf{C}^3 \setminus \{0\}})_* \mathcal{O}_\Gamma.$$

Using (5.12) as well as the push-pull formula $(\mathcal{F} \otimes f_* \mathcal{G}) = f_*(f^* \mathcal{F} \otimes \mathcal{G})$, we have

$$\begin{aligned}
j^* \circ \Phi_\theta(-) &= j^* \circ q_* (p^*(- \otimes \rho_0) \otimes \mathcal{U}_\theta) \\
&= t_* ((\text{id}_{\mathcal{M}_\theta} \times j)^* p^*(- \otimes \rho_0) \otimes (\text{id}_{\mathcal{M}_\theta} \times j)^* \mathcal{U}_\theta) \\
&= t_* ((\text{id}_{\mathcal{M}_\theta} \times j)^* p^*(- \otimes \rho_0) \otimes (i \times \text{id}_{\mathbf{C}^3 \setminus \{0\}})_* \mathcal{O}_\Gamma) \\
&= t_*(i \times \text{id}_{\mathbf{C}^3 \setminus \{0\}})_* ((i \times j)^* p^*(- \otimes \rho_0) \otimes \mathcal{O}_\Gamma) \\
&= s_*(r^* i^*(- \otimes \rho_0) \otimes \mathcal{O}_\Gamma) \\
&= \Phi_{\mathcal{O}_\Gamma} \circ i^*(-).
\end{aligned}$$

This concludes the proof. \square

Before we embark on the proof of (5.9), it is useful to recall some basic properties of local cohomology, see, e.g., [Har77, Chapter III, Exercise 2.3]. Let D be a closed subset of X and let \mathcal{E} be a sheaf on X . Denote by $\Gamma_D(X, \mathcal{E})$ the subspace of $\Gamma(X, \mathcal{E})$ consisting of sections whose support is contained in D . The functor $\Gamma_D(X, -)$ is left-exact. Then $H_D^\bullet(X, \mathcal{E}) := R^\bullet \Gamma_D(X, \mathcal{E})$ is called the *local cohomology* of \mathcal{E} with respect to D . Local cohomology is related to the usual cohomology of \mathcal{E} by the following long exact sequence

$$\cdots \rightarrow H_D^i(X, \mathcal{E}) \rightarrow H^i(X, \mathcal{E}) \rightarrow H^i(X \setminus D, \mathcal{E}|_{X \setminus D}) \xrightarrow{\delta} H_D^{i+1}(X, \mathcal{E}) \rightarrow \cdots$$

Moreover, it satisfies *excision*, that is, if U is an open subset in X containing D , then there is natural isomorphism

$$H_D^\bullet(X, \mathcal{E}) \cong H_D^\bullet(U, \mathcal{E}|_U).$$

Proof of Proposition 5.8. We already reduced the proof of this to the proof of the vanishing (5.9). Since by Proposition 5.10 $H^1(\mathcal{M}_\theta, \mathcal{E}nd(\mathcal{R})) = 0$, the long exact sequence associated to the local cohomology yields

$$H^0(\mathcal{M}_\theta, \mathcal{E}nd(\mathcal{R})) \xrightarrow{\delta} H_D^1(\mathfrak{M}_\theta, \mathcal{E}nd(\mathcal{R})(-D)) \rightarrow H^1(\mathfrak{M}_\theta, \mathcal{E}nd(\mathcal{R})(-D)) \rightarrow 0.$$

We show that the first map in this sequence is an isomorphism. This gives the desired vanishing, $H^1(\mathfrak{M}_\theta, \mathcal{E}nd(\mathcal{R})(-D)) = 0$.

Let H denote the hyperplane section in \mathbf{P}^3 . By excision, we have

$$\begin{aligned} H_D^1(\bar{\mathfrak{M}}_\theta, \mathcal{E}nd(\bar{\mathcal{R}})(-D)) &\cong H_D^1(\bar{\mathfrak{M}}_\theta \setminus \pi^{-1}(0), \mathcal{E}nd(\bar{\mathcal{R}})(-D)) \\ &= H_H^1([\mathbf{P}^3 \setminus \{[0:0:0:1]\}/G, \mathcal{O}(-1) \otimes \text{End}(R)) \\ &\cong H_H^1([\mathbf{P}^3/G], \mathcal{O}(-1) \otimes \text{End}(R)). \end{aligned}$$

Here and in the following we omit to make the appropriate restriction of sheaves explicit, when confusion is unlikely to arise. Using the above, we have the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{M}_\theta \setminus \pi^{-1}(0), \mathcal{E}nd(\mathcal{R})) & \xrightarrow{\delta} & H_D^1(\bar{\mathfrak{M}}_\theta, \mathcal{E}nd(\bar{\mathcal{R}})(-D)) \\ \downarrow \pi_* & & \downarrow \cong \\ H^0(\mathbf{C}^3 \setminus \{0\}/G, \mathcal{O}_{\mathbf{C}^3} \otimes \text{End}(R)) & \longrightarrow & H_H^1([\mathbf{P}^3/G], \mathcal{O}(-1) \otimes \text{End}(R)). \end{array}$$

We compose on the left with the commutative diagram in Proposition 5.10. Since $\delta = \tilde{\delta} \circ i^*$, we obtain the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{M}_\theta, \mathcal{E}nd(\mathcal{R})) & \xrightarrow{\delta} & H_D^1(\bar{\mathfrak{M}}_\theta, \mathcal{E}nd(\bar{\mathcal{R}})(-D)) \\ \Phi_\theta \downarrow & & \downarrow \cong \\ H^0(\mathbf{C}^3, \mathcal{O}_{\mathbf{C}^3} \otimes \text{End}(R))^G & \longrightarrow & H_H^1([\mathbf{P}^3/G], \mathcal{O}(-1) \otimes \text{End}(R)). \end{array}$$

All the vertical arrows are isomorphisms. Moreover, by using the long exact sequence associated to the local cohomology and

$$H^i([\mathbf{P}^3/G], \mathcal{O}(-1) \otimes \text{End}(R)) = H^i(\mathbf{P}^3, \mathcal{O}(-1) \otimes \text{End}(R))^G = 0$$

for $i = 0$ and 1 , it follows that the bottom map is also an isomorphism. Therefore the map δ must be an isomorphism. \square

6 Dirac operators on M_θ

Let (X, g) be an ALE spin manifold asymptotic to \mathbf{C}^n/G and let E be a complex vector bundle over X together with an asymptotically flat connection A . Denote by S^\pm be the spinor bundles on X and by D_E^\pm the corresponding twisted Dirac operators. We denote by $W_\delta^{k,2}(X, S^\pm \otimes E)$ the completions of the spaces of compactly supported sections with respect to the weighted Sobolev norm defined by (4.8). Let $D_{E,\delta}^\pm: W_\delta^{k+1,2}(X, S^\pm \otimes E) \rightarrow W_{\delta-1}^{k,2}(X, S^\mp \otimes E)$ denote the corresponding completion of the Dirac operator D_E^\pm .

Theorem 6.1. *For $\delta \in (-2n-1, 0)$, $D_{E,\delta}^\pm$ is Fredholm and its index is given by*

$$(6.2) \quad \text{index } D_{E,\delta}^+ = \int_X \text{ch}(E) \hat{A}(X) - \frac{\eta_E}{2}.$$

Here $\eta_E(s) := \sum_{\lambda \neq 0} \text{sign}(\lambda) |\lambda|^{-s}$ is the eta-function of the spectrum of the Dirac operator restricted to the boundary at infinity S^{2n-1}/G of the ALE manifold X , and $\eta_E(0)$ is the eta-invariant. Also $\hat{A}(X)$ is the Hirzebruch \hat{A} -polynomial applied to the Pontrjagin forms $p_i(X)$ of the ALE metric on X .

Proof. The fact that $D_{E,\delta}^\pm$ is Fredholm is proved as in Proposition 4.9 by noting that the set of indicial roots does not intersect $(-2n-1, 0)$. This can be seen, for example, by realising that the indicial roots correspond to the eigenvalues of the Dirac operator on S^{2n-1}/G shifted by $-\frac{2n-1}{2}$. The index formula follows from Atiyah–Patodi–Singer index theorem [APS75]. \square

We consider the case when X is a Calabi–Yau manifold and E underlies a holomorphic vector bundle \mathcal{E} with a Hermitian metric h whose induced Chern connection is A . Then there exists a canonical spin structure on X with

$$S^+ = \Lambda^{0,\text{even}} T_{\mathbb{C}}^* X \quad \text{and} \quad S^- = \Lambda^{0,\text{odd}} T_{\mathbb{C}}^* X.$$

The corresponding twisted Dirac operator is

$$(6.3) \quad D_{\mathcal{E}}^\pm = \sqrt{2} (\bar{\partial} + \bar{\partial}_h^*).$$

Now, we take X to be M_θ for $\theta \in \Theta_{\mathbf{Q}}$ generic and \mathcal{E} to be \mathcal{R} . Using Theorem 1.8, we equip M_θ with an ALE Calabi–Yau metric $g_{\theta,\text{RF}}$ and \mathcal{R} with an HYM metric.

Proposition 6.4. *The operator $D_{\mathcal{E}nd(\mathcal{R}),\delta}^+$ is an isomorphism for all $\delta \in (-5, 0)$. In particular, we have*

$$(6.5) \quad \text{index } D_{\mathcal{R}_\rho \otimes \mathcal{R}_\sigma, \delta}^+ = 0,$$

for all $\rho, \sigma \in \text{Irr}(G)$.

Proof. We prove that $\text{index } D_{\mathcal{E}nd(\mathcal{R}),\delta}^+ = 0$. Let Ω be a nowhere vanishing holomorphic volume form on M_θ , and $*$: $\Lambda^{p,q} T^* X \otimes \mathcal{E} \rightarrow \Lambda^{3-p,3-q} T^* X \otimes \mathcal{E}^*$ the Hodge $*$ -operator for some holomorphic bundle \mathcal{E} . Then, we have an isomorphism of vector bundles $S^+ \otimes \mathcal{E} \cong S^- \otimes \mathcal{E}^*$, given by

$$S^+ \otimes \mathcal{E} = \Lambda^{0,\text{even}} T^* M_\theta \otimes \mathcal{E} \xrightarrow{\Omega \wedge^-} \Lambda^{3,\text{even}} T^* M_\theta \otimes \mathcal{E} \xrightarrow{*} \Lambda^{0,\text{odd}} T^* M_\theta \otimes \mathcal{E}^* = S^- \otimes \mathcal{E}^*.$$

Similarly, $S^- \otimes \mathcal{E} \cong S^+ \otimes \mathcal{E}^*$. Via this identification $D_{\mathcal{E}}^+$ corresponds to $D_{\mathcal{E}^*}^-$. Thus $\text{index } D_{\mathcal{E},\delta}^+ = \text{index } D_{\mathcal{E}^*,\delta}^- = -\text{index } D_{\mathcal{E}^*, -5-\delta}^-$. Hence, in the case when $\mathcal{E} = \mathcal{E}nd(\mathcal{R})$, taking $\delta = -\frac{5}{2}$, it follows that $\text{index } D_{\mathcal{E}nd(\mathcal{R}), -\frac{5}{2}}^+ = 0$. Since the index is constant for $\delta \in (-5, 0)$, we must have $\text{index } D_{\mathcal{E}nd(\mathcal{R}),\delta}^+ = 0$ for all $\delta \in (-5, 0)$.

To complete the proof, we show that $\text{coker } D_{\mathcal{E}nd(\mathcal{R}),\delta}^+ = 0$ for $\delta \in (-5, 0)$. Since $\text{coker } D_{\mathcal{E}nd(\mathcal{R}),\delta}^+ \cong \ker D_{\mathcal{E}nd(\mathcal{R}), -5-\delta}^-$, it is enough to prove the last is trivial. Let $(\phi_1, \phi_3) \in \ker D_{\mathcal{E}nd(\mathcal{R})}^-$. Then we have $\bar{\partial}_h^* \phi_1 = 0$ and $\bar{\partial} \phi_1 + \bar{\partial}_h^* \phi_3 = 0$. The second identity gives

$$(6.6) \quad \bar{\partial} \bar{\partial}_h^* \phi_3 = 0.$$

Arguing as in Proposition 5.5 shows that $\phi_3 = O(r^{-4})$ and $\bar{\partial}_h^* \phi_3 = O(r^{-5})$. Hence, taking the L^2 -inner-product with ϕ_3 in (6.6) shows that $\bar{\partial}_h^* \phi_3 = 0$. Using the Kähler identity $\partial_A = i[\Lambda, \bar{\partial}_h^*]$ where Λ is the adjoint of the wedge multiplication by the Kähler form, yields $\partial_A \phi_3 = 0$. If we write $\phi_3 = \bar{\Omega} \otimes s$ where Ω is a holomorphic volume form on X and s is a smooth section of $\mathcal{E}nd(\mathcal{R})$, then $\partial_A(\bar{\Omega} \otimes s) = \bar{\Omega} \wedge \partial_A s = 0$ and thus $\partial_A s = 0$. Since the metric on \mathcal{R} is HYM, the Weitzenböck formula takes the form $2\Delta_{\partial_A} = \nabla_A^* \nabla_A$. Therefore, it follows that s is parallel and must hence vanish, as it decays at infinity. This implies that ϕ_1 satisfies $\bar{\partial} \phi_1 = 0$ and $\bar{\partial}^* \phi_1 = 0$. Then, by Lemma 5.1, ϕ_1 must vanish identically. \square

7 Geometrical McKay correspondence

In this section we prove Theorem 1.3 and derive its consequences. The proof uses the Atiyah–Patodi–Singer index theorem for ALE manifolds (6.2). In order to apply it, we need to compute the eta-invariant term that appears in this formula.

Proposition 7.1. *Let G be a finite subgroup of $\mathrm{SL}(n, \mathbf{C})$ acting freely on $\mathbf{C}^n \setminus \{0\}$. Assume that X is a smooth ALE spin manifold asymptotic to \mathbf{C}^n/G and let (E, A) be a asymptotically flat bundle on X whose fiber at infinity is E_∞ . Then, the eta-invariant for the Dirac operator $D_{E,\delta}$ on X is given by*

$$(7.2) \quad \eta_E = -\frac{2}{|G|} \sum_{\substack{g \in G \\ g \neq e}} \frac{\chi_{E_\infty}(g)}{\sum_{i=0}^n (-1)^i \chi_{\Lambda^i \mathbf{C}^n}(g)},$$

provided $-2n + 1 < \delta < 0$. In this formula χ_{E_∞} denotes the character of the representation corresponding to the action of G on the vector space E_∞ .

Remark 7.3. Note that for any $g \in \mathrm{SL}(n, \mathbf{C})$, $\sum_{i=0}^n (-1)^i \chi_{\Lambda^i \mathbf{C}^n}(g) = \det(\mathrm{id} - g)$. Since G is chosen to act freely on \mathbf{C}^n , $\det(I_n - g) \neq 0$ for all $g \in G \setminus \{e\}$, and thus all the denominators in formula (7.2) are non-zero.

This proposition is a consequence of the Lefschetz fixed-point formula, in the sense that η_E is the contribution from the fixed locus under the action of G on \mathbf{C}^n . It can be also proved using the definition of the eta-invariant as the analytic continuation at 0 of the eta-series corresponding to the spectrum of the Dirac operator on the boundary at infinity of the orbifold \mathbf{C}^n/G . This last approach gives the generalization of the above formula to the case of non-isolated singularities [Deg01].

Consider the virtual representation $\sum_{i=0}^n (-1)^i \Lambda^i \mathbf{C}^n$ of G . For each $\rho \in \mathrm{Irr}(G)$ we have the decomposition into irreducibles

$$(7.4) \quad \Lambda^i \mathbf{C}^n \otimes \rho = \sum_{\sigma \in \mathrm{Irr}(G)} a_{\rho\sigma}^{(i)} \sigma.$$

We define

$$(7.5) \quad c_{\rho\sigma} := \sum_{i=0}^n (-1)^i a_{\rho\sigma}^{(i)}.$$

Let \tilde{C} be the matrix with entries $c_{\rho\sigma}$ for $\rho, \sigma \in \mathrm{Irr}(G)$, and let C be the principal submatrix of \tilde{C} obtained by erasing the line and column corresponding to the trivial representation.

When G is a finite subgroup of $\mathrm{SL}(2, \mathbf{C})$, the matrix C is the Cartan matrix of the unique simple Lie algebra corresponding to G , while \tilde{C} is the extended version. This is the essence of the McKay correspondence [McK80]. Note that for $n \geq 3$ this matrix is not the Cartan matrix associated to a Lie algebra, nor is it a generalised Cartan matrix as appearing in the context of Lie algebras.

When n is even, the virtual representation $\sum_{i=0}^n (-1)^i \Lambda^i \mathbf{C}^n$ is self-dual, while when n is odd, it is anti-self-dual. Hence C is a symmetric matrix when n is even, and skew-symmetric when n is odd. Moreover, note that when G is abelian (in particular, when G acts freely on $\mathbf{C}^n \setminus \{0\}$), all the irreducible representations are one-dimensional, and thus $\sum_{\sigma \in \mathrm{Irr}(G)} a_{\rho\sigma}^{(i)} = \dim \Lambda^i \mathbf{C}^n$. Combined with formula (7.5), this gives

$$(7.6) \quad \sum_{\sigma \in \mathrm{Irr}(G)} c_{\rho\sigma} = 0$$

for all $\rho \in \text{Irr}(G)$.

Now we are ready to prove our main result Theorem 1.3. The Chern character we are using is $\tilde{\text{ch}} := \text{ch} - \text{rk}$, the Chern character from the reduced topological K -theory $\tilde{K}(X) = \ker(\text{rk}: K(X) \rightarrow \mathbf{Z})$ from $\bigoplus_{k \geq 1} H^k(X, \mathbf{R})$.

Proof of Theorem 1.3. Let X be a projective crepant resolution of \mathbf{C}^3/G . By Theorem 2.13, X is the moduli space \mathcal{M}_θ of θ -stable G -constellations for some $\theta \in \Theta_{\mathbf{Q}}$ generic. Using Proposition 3.5, the analytification of \mathcal{M}_θ is the smooth Kähler quotient M_θ . We can thus assume that $X = M_\theta$.

Let $\text{Irr}(G)$ be the set of irreducible representations of G , ρ_0 be the trivial representation and $\text{Irr}_0(G)$ be the set of non-trivial irreducible representations. Note that since G acts freely on $\mathbf{C}^3 \setminus \{0\}$, the group G must be abelian and, hence, all its irreducible representations are one dimensional.

By Proposition 6.4, we know that $\text{index } D_{\mathcal{R}_\rho \otimes \mathcal{R}_\sigma^*}^+ = 0$ for all weights $\delta \in (-5, 0)$ and for all $\rho, \sigma \in \text{Irr}(G)$. Then (6.2) gives

$$(7.7) \quad \int_{M_\theta} \text{ch}(\mathcal{R}_\rho \otimes \mathcal{R}_\sigma^*) \hat{A}(M_\theta) = \frac{\eta_{R_\rho \otimes R_\sigma^*}}{2}.$$

Multiplying on the right by the matrix \tilde{C} , we obtain

$$(7.8) \quad \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \int_{M_\theta} \text{ch}(\mathcal{R}_\rho \otimes \mathcal{R}_\sigma^*) \hat{A}(M_\theta) = \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \frac{\eta_{R_\rho \otimes R_\sigma^*}}{2}$$

for all $\tau \in \text{Irr}(G)$. The left-hand-side of (7.8) can be written as

$$\begin{aligned} & \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \int_{M_\theta} \tilde{\text{ch}}(\mathcal{R}_\rho) \tilde{\text{ch}}(\mathcal{R}_\sigma^*) \\ & + \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \int_{M_\theta} \text{ch}(\mathcal{R}_\rho) \hat{A}(M_\theta) + \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \int_{M_\theta} \text{ch}(\mathcal{R}_\sigma^*) \hat{A}(M_\theta) \\ & = \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \int_{M_\theta} \tilde{\text{ch}}(\mathcal{R}_\rho) \tilde{\text{ch}}(\mathcal{R}_\sigma^*) + \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \frac{\eta_{R_\rho}}{2} + \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \frac{\eta_{R_\sigma^*}}{2} \end{aligned}$$

for all $\tau \in \text{Irr}(G)$. Note that we have used the fact that, since M_θ has complex dimension 3, $\hat{A}(M_\theta) = 1 + \left(\hat{A}(M_\theta)\right)_4$. Moreover, by (7.6) the third term above vanishes. Therefore

$$(7.9) \quad \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \int_{M_\theta} \tilde{\text{ch}}(\mathcal{R}_\rho) \tilde{\text{ch}}(\mathcal{R}_\sigma^*) = \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \frac{\eta_{R_\rho \otimes R_\sigma^*}}{2} - \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \frac{\eta_{R_\rho}}{2}.$$

Taking the characters of (7.4) and summing over i with alternating signs, we obtain

$$(\chi_{\mathbf{C}^3}(g) - \chi_{\Lambda^2 \mathbf{C}^3}(g)) \chi_\tau(g) = - \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \chi_\rho(g).$$

This gives

$$\chi_{\tau \otimes \sigma^*}(g) = - \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \frac{\chi_{\rho \otimes \sigma^*}(g)}{\chi_{\mathbf{C}^3}(g) - \chi_{\Lambda^2 \mathbf{C}^3}(g)}$$

for all $g \in G \setminus \{e\}$. Summing over all such g and using formula (7.2) for the eta-invariant, we obtain

$$(7.10) \quad -2 \left(\delta_{\tau\sigma} - \frac{1}{|G|} \right) = \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \eta_{\mathcal{R}_\rho \otimes \mathcal{R}_\sigma^*}$$

Hence, (7.9) yields

$$\sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \int_{M_\theta} \tilde{\text{ch}}(\mathcal{R}_\rho) \tilde{\text{ch}}(\mathcal{R}_\sigma^*) = - \left(\delta_{\tau\sigma} - \frac{1}{|G|} \right) - \left(\delta_{\tau\rho_0} - \frac{1}{|G|} \right)$$

for all $\tau, \sigma \in \text{Irr}(G)$. In particular, for all $\tau \in \text{Irr}_0(G)$, this becomes

$$(7.11) \quad \sum_{\rho \in \text{Irr}(G)} c_{\tau\rho} \int_{M_\theta} \tilde{\text{ch}}(\mathcal{R}_\rho) \tilde{\text{ch}}(\mathcal{R}_\sigma^*) = -\delta_{\tau\sigma}.$$

Since $\text{ch}(\mathcal{R}_{\rho_0}) - 1 = 0$, it follows that the matrix $C = (c_{\tau\rho})_{\tau, \rho \in \text{Irr}_0(G)}$ is invertible and

$$\int_{M_\theta} \tilde{\text{ch}}(\mathcal{R}_\rho) \tilde{\text{ch}}(\mathcal{R}_\sigma^*) = - (C^{-1})_{\rho\sigma},$$

for all $\rho, \sigma \in \text{Irr}_0(G)$, which is precisely (1.4). \square

Remark 7.12. Note that formula (7.11) gives that the matrix C is invertible. In the case of a finite subgroup of $\text{SL}(2, \mathbf{C})$, the invertibility of C was a direct consequence of the McKay Correspondence, given that C is the Cartan matrix associated to a simply-laced Dynkin diagram [McK80].

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